Technical Notes

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Vortex and Momentum Theories for Hovering Rotors

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MILLER[1] has called attention to a seeming paradox in the vortex theory for hovering rotors. In the usual application of vortex theory for the limiting case of an infinite number of rotor blades to propellers, the induced velocities in the wake are small compared to the forward velocity and the distributed vortices in the wake may be assumed to move downstream at the propeller forward velocity for lightly loaded propellers. Theodorsen showed,[2] however, that in the application of the Goldstein theory for optimum circulation distribution on propellers with a finite number of blades, it is necessary, for more heavily loaded propellers representative of practical designs, to take into account the induced velocities in arriving at vortex geometry in the wake. Theodorsen also gave an approximate method for computing slipstream contraction, but for the cases of propellers in cruise flight which he considered, the slipstream contraction was of the order of 1%; the analysis specifically excluded static thrust conditions.

For propeller cruise operating conditions it is well known[3,4] that the results of vortex theory for an infinite number of blades are essentially identical to the momentum theory of the actuator disk; this remains true for the case of nonconstant blade circulation if both theories are applied to differential annular strips on the blades. It is also true when induced wake rotation is introduced into both theories. A particular result of practical significance in the theories (neglecting wake rotation) is that the induced axial velocity at the propeller v_I is one-half the velocity in the ultimate wake, v_2. In the propeller case, with negligible slipstream contraction, this follows from the fact that the vortices are assumed to be uniformly distributed over the circumference of a cylinder of approximately constant diameter that has a semi-infinite length viewed from the plane of the propeller and an infinite length viewed from a transverse plane in the ultimate wake, giving rise to a factor of two in the calculated induced velocities. It should be noted, however, that in the momentum theory of the actuator disk it is assumed that the flow velocity in the wake is constant for constant disk loading. In the vortex theory, this is a conclusion arrived at from analyses of the induced flowfield in the vortex cylinder. (Yet another approach is to represent the acutator disk by a doublet layer; this gives results equivalent to vortex theory.)

With large slipstream contractions, such as occur for the static thrust condition of a propeller or for a hovering helicopter rotor, it is no longer valid to make this particular calculation since the wake vortices cannot be considered to lie on cylinders of constant diameter in the vicinity of the propeller plane. Yet, even in this case, vortex theory for an infinite number of blades and constant disk loading leads to the result, $\bar{v}_I = ½ v_2$, if induced rotation in the wake is neglected, where $\bar{v}_I$ is the average velocity at the rotor. Miller noted that for the static thrust case of a rotor, the calculation of the velocity induced at the rotor by a vortex cylinder, assumed to have a constant diameter, leads to the same result as the actuator disk momentum theory only if an incorrect vortex displacement velocity is used.

The apparent paradox in the application of vortex theory to hovering rotors indicated by Miller can be resolved by recognizing that the assumption of cylindrical shape of vortex sheets is only valid in the ultimate wake (corresponding to the Trefftz plane in wing theory). For blades of constant circulation (or constant thrust over the rotor disk for an infinite number of blades) the thrust T is given by

$$T = Q\int_0^R \rho\Omega r\Gamma \mathrm{d}r$$

where R is the rotor radius, Q the number of blades, Ω the angular velocity of the rotor, ρ the air density, and Γ is the circulation on a blade, or

$$T = \rho Q(R^2/2)\Omega\Gamma$$

The thrust per unit area is given by

$$T/A_I = \rho Q\Omega\Gamma/2\pi \tag{1}$$

where A_I is the rotor area.

The strength per unit length γ of the elemental ring vortices comprising the vortex cylinder in the approximation of an infinite number of blades is given by

$$\gamma = Q\Gamma/h$$

where h is the distance traveled by the wake vortices per rotor revolution or

$$h = 2\pi v_2/2\Omega$$

Here the velocity at the vortex cylinder has been taken to be one-half of the velocity inside the wake, v_2, in line with the usual theory of free vortices and vortex sheets and as indicated by Miller in his discussion of the velocity field of a vortex cylinder. Thus

$$\gamma = 2Q\Omega\Gamma/2\pi v_2 \tag{2}$$

In the far downstream wake, the velocity induced internal to a vortex cylinder of infinite extent is $v_2 = \gamma$, and is uniform over the wake. Thus

$$v_2 = 2Q\Omega\Gamma/2\pi v_2 \tag{3}$$

Substituting from Eq. (1)

$$v_2^2 = 2T/\rho A_I \tag{4}$$

This result is the same as that usually obtained in the momentum theory of the actuator disk by calculating the momentum balance between planes just ahead of and just behind the rotor disk and using Bernoulli's theorem for the upstream and downstream sides of the disk. In effect, the total head is increased by T/A_I in passage of the fluid through the actuator disk. The mechanism of lift generation is left

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Dedicated to Professor W. R. Sears on the occasion of his 70th birthday with the author's congratulations and best wishes.

unstated in the actuator disk theory, whereas the result in vortex theory follows directly from the classical vortex theory of lift for airfoils.

To complete the analysis we note that the thrust of the rotor must equal the rate of momentum flow in the wake or

$$T=\rho A_2 v_2^2$$

where A_2 is the cross-sectional area of the wake cylinder far downstream of the rotor. Combining this with Eq. (4) leads to

$$A_1=2A_2 \quad (5)$$

Applying the equation of continuity between the rotor plane and a plane in the cross section of the ultimate wake leads to

$$\bar{v}_1=v_2/2 \quad (6)$$

where $\bar{v}_1$ is the average axial velocity over the rotor disk.

The results of vortex theory for an infinite number of blades in Eqs. (4) and (6) are in agreement with the momentum theory of the actuator disk for the hovering rotor just as they are for the propeller in forward flight. This analysis can be extended to nonconstant blade circulation and also leads to results similar to those for the annular momentum theory of the actuator disk. Further, rotational flow in the wake can be treated by vortex theory for the hovering rotor by considering the flow induced by the line vortex of strength $Q\Gamma$ shed along the axis of the cylinder, with results again the same as the actuator-disk momentum theory.

For nonconstant blade circulation both the momentum theory of the actuator disk and the vortex theory for an infinite number of blades require in this formulation additional assumptions regarding the independent applicability to differential annular stream tubes of the conservation of momentum; the range of validity or accuracy of this assumption has not been analytically determined. However, the vortex theory for an infinite number of blades and the momentum theory of the actuator disk for the hovering rotor with nonconstant blade circulation do not differ with regard to this assumption. In fact, Eq. (2) simply becomes

$$\frac{d\gamma}{dr}=2\frac{Q\Omega}{2\pi v_2}\frac{d\Gamma}{dr} \quad (7)$$

giving, in place of Eq. (3),

$$\frac{d}{dr}\left(\frac{v_2^2}{2}\right)=\frac{Q\Omega}{2\pi}\frac{d\Gamma}{dr} \quad (8)$$

The thrust loading per unit area, t, averaged over the rotor circumference at radius r is given by

$$t=\rho(\Omega Q\Gamma/2\pi) \quad (9)$$

Thus, integrating Eq. (8) with $\Gamma=0$ at the tip we obtain

$$v_2^2/2=t/\rho \quad (10)$$

This corresponds to the differential form of Eq. (4).

In this theory for nonconstant blade circulation the single cylinder wrapped by vortices springing from the propeller tip is replaced by a continuum of cylinders, each fed by vortices of strength $d\Gamma/dr$ shed at radius r in the plane of the rotor. That this theory gives the correct induced velocity at the vortex sheet may most easily be seen from Eq. (8) for a finite jump in vortex strength, $\Delta\Gamma$, at some radius R_1 corresponding to the shedding by the rotor blade of a discrete vortex cylinder in addition to the vortex cylinder assumed to exist at the tip $(r=R)$, inside of which the constant increment of induced axial velocity in the wake is v_{2a}. The additional vortex cylinder at $r=R_1$ induces inside itself an additional constant axial velocity, v_{2b}. Integrating Eq. (8) in a Stieltjes sense results in

$$\Delta\left(\frac{v_2^2}{2}\right)=\frac{(v_{2a}+v_{2b})^2}{2}-\frac{v_{2a}^2}{2}=\frac{Q\Omega}{2\pi}(\Delta\Gamma)_{R_1}$$

or

$$v_{2b}=\frac{Q\Omega}{2\pi[v_{2a}+(v_{2b}/2)]}(\Delta\Gamma)_{R_1}$$

The velocity appearing in the denominator of the term on the left is the mean axial velocity in the ultimate wake of the cylindrical vortex sheet springing from $r=R_1$, which is appropriate for determining the axial intensity of the vortex sheet in the wake according to Eq. (7), remembering that a discrete vortex cylinder of intensity γ leads to an axial velocity in the ultimate wake inside it of $v_2=\gamma$ and zero outside. Thus, again, for the annular differential momentum theory of the actuator disk under the usual assumptions[3,4] there is an equivalence with vortex theory for similar physical models.

The exact equivalence of the actuator disk momentum theory an annular differential form and vortex theory for rotors has been questioned by Johnson,[5] who includes a thorough and careful presentation of the various theories for flows through propellers and helicopter rotors. The particular case for which this question was examined by Johnson was the propeller in forward flight or the rotor in vertical climb. In this case there is an imposed velocity V relative to the plane of the blades or the actuator disk. Then the additional velocity V is imposed on the motion of all vortex sheets and Eq. (8) becomes

$$\frac{d}{dr}\frac{(V+v_2)^2}{2}=\frac{Q\Omega}{2\pi}\frac{d\Gamma}{dr} \quad (11)$$

This result again corresponds exactly to the annular momentum theory of the actuator disk. Johnson appears to have overlooked the exact differential relationship, Eq. (11), and proceeded to a representation of v_2 by integration of parts, which, while correct, obscured the equivalence between the two theories.

Thus it has been shown that vortex theory can be unambiguously applied to the hovering rotor and, for the case of an infinite number of blades, is substantially equivalent to the momentum theory of the actuator disk. However, this does not detract from the value of simplified models of the vortex wake such as those discussed by Miller,[1] since such models can often provide valuable insights into aspects of vortex interaction with the rotor which are not easily achieved through detailed numerical computations with more complete vortex theories for a finite number of blades.

References

[1] Miller, R. H., "Vortex Theory for Hovering Rotors," *AIAA Journal*, Vol. 20, Dec. 1982, pp. 1754-1756.

[2] Theodorsen, T., *Theory of Propellers*, McGraw-Hill Book Co., New York, 1948, pp. 6-7.

[3] Glauert, H., "Airplane Propellers," *Aerodynamic Theory*, Vol. IV, edited by W. E. Durand, Durand Reprinting Comm. California, 1943, pp. 230-235.

[4] Glauert, H., *The Elements of Airfoil and Airscrew Theory*, Cambridge University Press, Cambridge, England, 1937, pp. 208-212.

[5] Johnson, W., *Helicopter Theory*, Princeton University Press, Princeton, N.J., 1980, pp. 79-81.